Topic :→ Integration by parts.
(Problem Contd)

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Problem :→ Integrate $\int [\log x]^2 dx$

Let $I = \int [\log x]^2 dx$

Since it is single logarithmic
we take function

So $f(x) = [\log x]^2 \qquad g(x) = 1$

$f'(x) = \frac{2\log x}{x} \qquad \int dx\, dx = x$

Now Integrating by parts.

$I = f(x) \cdot \int g(x)\, dx - \int [\int g(x) dx]\, f'(x)\, dx$

$= x[\log x]^2 - 2\int \frac{x \log x}{x} dx$

$= x[\log x]^2 - 2 I_1$

Here $I_1 = \int \log x\, dx$

$= \log x \int dx - \int [\int dx] \frac{d \log x}{dx} \cdot dx$

$= x\log x - \int x \cdot \frac{1}{x} dx$

$= x\log x - x$

Thus $I = (x\log x)^2 - 2(x\log x - x) + C$

$= (x\log x)^2 - 2x\log x + 2x + C$

Problem :→ Integrate $\int x^2 \tan^{-1} x\, dx$

Let $I = \int x^2 \tan^{-1}x \, dx$

1st Method :->

Here $f(x) = \tan^{-1}x$ $g(x) = x^2$

$f'(x)\,dx = \frac{1}{1+x^2}$ $\int g(x)\,dx = \frac{x^3}{3}$

Using Method of Integration by Parts—

$$I = \tan^{-1}x \int x^2 dx - \int\left(\int x^2 dx\right) \frac{d \tan^{-1}x}{dx} dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x^3}{1+x^2} dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x \cdot x^2}{1+x^2} dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3} I_1$$

$$I_1 = \int \frac{x \cdot x^2}{1+x^2} dx$$ Put $x = \tan\theta$.

$dx = \sec^2\theta \, d\theta$

$x^2 = \tan^2\theta = \sec^2\theta - 1$

$1+x^2 = 1+\tan^2\theta$
$= \sec^2\theta$

$$= \int \frac{\cancel{\tan\theta\,(\sec^2\theta)}}{}$$

$$\int \frac{\tan\theta\,(\sec^2\theta - 1)}{\sec^2\theta} \sec^2\theta \, d\theta$$

$$= \int \tan\theta \sec^2\theta \, d\theta - \int \tan\theta \, d\theta$$

$$= \frac{\tan^2\theta}{2} - \log\sec\theta$$

$$= \frac{x^2}{2} - \log\sqrt{1+x^2}$$

$\therefore\ I = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\left[\frac{x^2}{2} - \log\sqrt{1+x^2}\right] + C$

$= \frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + \frac{1}{3}\log\sqrt{1+x^2} + C$

$= \frac{x^3}{3}\tan^{-1}y - x^3/6 + \frac{1}{6}\log(1+x^2) + C$

Problem 2 → Integrate $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Here. Let $I = \int \frac{x.\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Put $\sin^{-1}x = z$

$\sin z = x$

$\frac{1}{\sqrt{1-x^2}}\,dx = dz$

$1-x^2 = \cos^2 z$

$\sqrt{1-x^2} = \cos z$

$= \int \frac{z\sin z}{\cos z}\cos z\,dz$

$= \int z\sin z\,dz$

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z $\sin z$

$= \int f(z).g(z)\,dz$ Where $F(z) = z$ $g(z) = \sin z$

Integrating by Parts

$\frac{df(z)}{dz} = 1$ $\int g(z)\,dz$

$I = f(z)\int g(z)\,dz - \int\left[\int g(z)\,dz\right]f'(z)\,dz$ $= \int \sin z\,dz$

$= -\cos z$

$= -z\cos z + \int \cos z\,dz$

$= -z\cos z + \sin z + C$

$= -\sqrt{1-x^2}\sin^{-1}x + x + C$

Problem 3. Integrate

$$\int \frac{x^2 \tan^{-1}x}{1+x^2}\, dx$$

Let $I = \int \frac{x^2 \tan^{-1}x\, dx}{1+x^2}$

Put $\tan^{-1}x = z$

$\Rightarrow \tan z = x$

$\Rightarrow \tan^2 z = x^2$

$\frac{1}{1+x^2}\, dx = dz$

$$= \int z \tan^2 z\, dz$$

$$= \int z(\sec^2 z - 1)\, dz$$

$$= \int z \sec^2 z\, dz - \int z\, dz$$

$$= I_1 - I_2 \quad \text{——— (1)}$$

$$I_1 = \int z \sec^2 z\, dz$$

$$= \int f(z)\, g(z)\, dz$$

Where $f(z) = z$

$g(z) = \sec^2 z$

Integrating by Parts

$\Rightarrow f'(z) = 1$

$\int g(z)\, dz = \int \sec^2 z\, dz = \tan z$

$$I_1 = f(z) \int g(z)\, dz - \int \left[\int g(z)\, dz\right] f'(z)\, dz$$

$$= z \tan z - \int \tan z\, dz$$

$$= z \tan z - \log \sec z$$

$$= x \tan^{-1}x - \log \sqrt{1+x^2}$$

$$= x \tan^{-1}x - \frac{1}{2}\log(1+x^2)$$

$$I_2 = \int z\,dz = \frac{z^2}{2} = \frac{(\tan^{-1}x)^2}{2}$$

$\therefore$ from (1)

$$I = I_1 - I_2$$

$$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2) - \frac{(\tan^{-1}x)^2}{2} + C$$

Problem 4. Integrate

$$\int \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}\,dx$$

Let $I = \int \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}\,dx$

Put $\tan^{-1}x = z \Rightarrow \tan z = x$

$\frac{1}{1+x^2}\,dx = dz$ $\quad 1+x^2 = 1+\tan^2 z$

$= \sec^2 z$

$\Rightarrow \sqrt{1+x^2} = \sec z.$

$$= \int \frac{z\tan z}{\sec z}\,dz$$

$(1+x^2)^{3/2} = (1+x^2)(1+x^2)^{1/2}$

$= \int z\sin z\,dz$ $\quad$ Here $f(z) = z$ $\quad g(z) = \sin z$

$\Rightarrow f'(z) = 1$ $\quad \int g(z)dz = \int \sin z\,dz$

$= -\cos z.$

Now Integrating by Parts

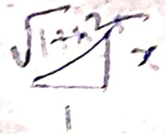
$$I = z\int \sin z\,dz - \int\left[\int \sin z\,dz\right]1.dz$$

$$= -z\cos z + \int \cos z\,dz$$

$$= -z\cos z + \sin z$$

Putting the value of z, $\cos z$ and $\sin z$

Since $\tan z = x$



$\Rightarrow \sin z = \dfrac{x}{\sqrt{1+x^2}}$

$\cos z = \dfrac{1}{\sqrt{1+x^2}}$

$$\therefore I = -\frac{1}{\sqrt{1+x^2}}\tan^{-1}x + \frac{x}{\sqrt{1+x^2}} + C$$

Integrate $\displaystyle\int \frac{\tan^{-1}x}{(1+x)^2}\,dx$

Let $\displaystyle I = \int \frac{\tan^{-1}x}{(1+x)^2}\,dx$

$$= \int \frac{\tan^{-1}x}{1+x^2+2x}\,dx$$

Put $\tan^{-1}x = z$

$\tan z = x$ $\qquad 1+x^2 = \sec^2 z$

$\sec^2 z\,dz = dx$ $\qquad 2x = 2\tan z$

$\sec^2 z\,dz = dx$

$$= \int \frac{z \cdot \sec^2 z\,dz}{\sec^2 z + 2\tan z}$$

$$= \int \frac{z\sec^2 z\,dz}{\sec^2 z\left(1 + \frac{2\tan z}{\sec^2 z}\right)}$$

$$= \int \frac{z}{1+2\sin z\cos z}\,dz$$

$$= \int \frac{z}{1+\sin 2z}\,dz$$

$$I = \int \frac{z}{1+\cos(2z-\pi/2)}\,dz$$

$$= \int \frac{z}{2\cos^2(z-\frac{\pi}{4})}\,dz$$

$$= \frac{1}{2}\int z\,(\sec^2(z-\frac{\pi}{4})\,dz$$

$$= \frac{1}{2}\int z\sec^2(z-\frac{\pi}{4})\,dz$$

$$= \frac{1}{2} I_1$$

$$I_1 = \int z \sec^2(z-\frac{\pi}{4})\,dz$$

Integrating by parts

$$= z\int \sec^2(z-\frac{\pi}{4})\,dz - \int\left[\int \sec^2(z-\frac{\pi}{4})dz\right]\frac{dz}{dz}\,.dz$$

$$= z\tan(z-\frac{\pi}{4}) - \int \tan(z-\frac{\pi}{4})\,dz$$

$$= z\tan(z-\pi/4) - \log\sec(z-\pi/4)$$

$$= \tan^{-1}x\,\cancel{\tan}\left(\frac{\tan z-1}{\tan z+1}\right) - \log\sqrt{1+\tan^2(z-\frac{\pi}{4})}$$

$$\therefore I = \frac{1}{2}\tan^{-1}x\,\cancel{\tan}\,\frac{x-1}{x+1} - \frac{1}{2}\log\left[1+\left(\frac{x-1}{x+1}\right)^2\right].$$

$$= \frac{1}{2}\tan^{-1}x\,\frac{x-1}{x+1} - \frac{1}{4}\log\left[1+\left(\frac{x-1}{x+1}\right)^2\right] + c$$